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ON THE STABILITY OF CERTAIN MOTIONS OF A RIGID BODY WITH ELASTIC RODS AND LIQUID

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We consider the motion of a free rigid body with three pairs of elastic rods and with a cavity containing a liquid, in two cases: in a central Newtonian force field and in the absence of external forces. By an application of Rumiantsev's theorem [1] we obtain sufficient stability conditions for relative equilibrium in a circular orbit and to uniform rotations of this system. We show that the presence of a liquid with a free surface in the cavity and the connection of elastic rods to the body have a destabilizing effect on the stability of the corresponding unperturbed motions of the unaltered system. We also point out sufficient stability conditions in the case when less than three pairs of rods are attached to the body. For a large Young's modulus the stability conditions obtained lead (in the absence of the liquid) to the well-known sufficient conditions for the stability of a rigid body. Stability conditions for the case when one pair of rods is attached to the body and when there is no liquid are compared with the stability conditions obtained in [2, 3]. In connection with the assertion made in [2, 3] regarding the novelty of the method used, we remark that this method was previously developed by Rumiantsev and was applied to the solution of a number of problems on the stability of the steady-state motions of a rigid body with a liquid filling [4].

1. We consider the motion in a central Newtonian force field of a rigid body having an arbitrarily-shaped cavity wholly or partially filled with a homogeneous incompressible ideal liquid and carrying a certain number of thin inextensible elastic rods each of which has a constant cross-section and two planes of symmetry. Neglecting the influence of the relative motion of the system on the motion of its center of mass, we take it that the latter moves uniformly along a circular Keplerian orbit with angular velocity Ω. We introduce right rectangular coordinate axes systems: an orbital one $Cxyz$ with origin at the center of mass C of the system and with axes directed along the tangent, the binormal, and the radius vector of the orbit, respectively, and an attached one $Ox_1x_2x_3$ with origin at the center of mass O of the rigid body and with axes directed along the axes of its central inertia ellipsoid. Let $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ be the unit vectors along the axes x_1, x_2, x_3. We denote the unit vectors of the y and z axes by $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$, and their projections onto the x_1, x_2, x_3 axes by $\beta_1, \beta_2, \beta_3$ and $\gamma_1, \gamma_2, \gamma_3$. These quantities are related by the equalities

$$\chi_1 \equiv \gamma_1^2 + \gamma_2^2 + \gamma_3^2 - 1 = 0, \qquad \chi_2 \equiv \gamma_1\beta_1 + \gamma_2\beta_2 + \gamma_3\beta_3 = 0$$
$$\chi_3 \equiv \beta_1^2 + \beta_2^2 + \beta_3^2 - 1 = 0 \tag{1.1}$$

We shall take it that one, two, or three pairs of elastic rods of length l have been fixed to the body at like distances a from the point O and, in the undeformed state, are situated along the axes x_1, x_2, x_3. Here the coordinate planes serve as the planes of symmetry of the rods. We use the indices 1 - 3 and 4 - 6 to identify the rods situated along the positive and the negative directions of the axes x_1, x_2, x_3 respectively. By

$$\mathbf{u}_j(s, t) = u_{1j}\mathbf{i}_1 + u_{2j}\mathbf{i}_2 + u_{3j}\mathbf{i}_3, \qquad 0 \leqslant s \leqslant l, \quad t \geqslant t_0 \qquad (j = 1, 2, \ldots, 6)$$

we denote the elastic displacement vector of the points of the axis of the j th rod. The condition that the rods are inextensible leads to the relations [5]

$$u_{11} = -{}^1/_2(u_{21}'^2 + u_{31}'^2), \quad u_{22}' = -{}^1/_2(u_{32}'^2 + u_{12}'^2), \quad u_{33}' = -{}^1/_2(u_{13}'^2 + u_{23}'^2) \tag{1.2}$$

$$u_{14}' = {}^1/_2(u_{24}'^2 + u_{34}'^2), \quad u_{25}' = {}^1/_2(u_{35}'^2 + u_{15}'^2), \quad u_{36}' = {}^1/_2(u_{16}'^2 + u_{26}'^2)$$

$$u' = \partial u/\partial s$$

The condition that the ends of the rods are fixed to the body leads to the boundary conditions

$$u_{ij} = u_{i,\,3+j} = 0, \quad u_{ij}' = u_{i,\,3+j}' = 0 \qquad (i, j = 1, 2, 3;\ i \neq j) \ \text{for}\ s = 0 \tag{1.3}$$

From (1.2) it follows that u_{ii}, $u_{i,3+i}$ $(i = 1,2,3)$ are quantities of the second order of smallness if as quantities of the first order of smallness we take u_{ij}, $u_{i,3+j}$ and their first derivatives u_{ij}', $u_{i,3+j}'$ $(i, j = 1, 2, 3;\ i \neq j)$. Note that equalities (1.2) represent the inextensibility condition for the rods only to within terms of the second order of smallness relative to the quantities indicated.

For the potential energy Π_d of elastic deformation we use the expression [5]

$$\Pi_d = \frac{1}{2} E \int_0^l (I_{31}u_{21}''^2 + I_{21}u_{3}''^2 + I_{12}u_{32}''^2 + I_{32}u_{12}''^2 + I_{23}u_{13}''^2 + I_{13}u_{23}''^2 + \tag{1.4}$$

$$+ I_{34}u_{24}''^2 + I_{24}u_{34}''^2 + I_{15}u_{35}''^2 + I_{35}u_{15}''^2 + I_{26}u_{16}''^2 + I_{13}u_{26}''^2)\, ds$$

Here E is the Young modulus, I_{ij} is the moment of inertia of the cross section of the j th rod relative to the straight line drawn through the center of gravity of the section parallel to the x_i-axis, EI_{ij} is the bending inflexibility. The position of any point of the system relative to the coordinate axes $Ox_1\,x_2\,x_3$ is determined by its radius vector $\mathbf{r}$. For the points of the rods $\mathbf{r} = \mathbf{r}^\circ + \mathbf{w}(\mathbf{r}^\circ, t)$, where $\mathbf{w}(\mathbf{r}^\circ, t)$ is the elastic displacement vector of the points of the rod, whose position upto the deformation is determined by the radius vector $\mathbf{r}^\circ$. The potential energy Π_g of the force of attraction, computed to within terms of order $L^3R_c^{-3}$, where L is the characteristic linear dimension of the body and R_c is the orbit's radius, is determined by the formula

$$\Pi_g = {}^1/_2\,\Omega^2(3\boldsymbol{\gamma}\cdot\boldsymbol{\Theta}^c\cdot\boldsymbol{\gamma} - \operatorname{sp}\boldsymbol{\Theta}^c)$$

where Θ^c is the system's energy tensor for the point C with the components

$$\vartheta_{11} = J_1 - M(x_{2c}^2 + x_{3c}^2) + \sigma\rho_1 \int_0^l \Big\{u_{21}^2 + u_{31}^2 + u_{24}^2 + u_{34}^2 + u_{32}^2 + u_{35}^2 + u_{23}^2 +$$

$$+ u_{26}^2 - \Big[a(l-s) + \frac{1}{2}(l^2 - s^2)\Big](u_{32}'^2 + u_{12}'^2 + u_{35}'^2 + u_{15}'^2 + u_{13}'^2 + u_{23}'^2 + u_{16}'^2 +$$

$$+ u_{26}'^2)\Big\}\, ds + \rho_2 \int_\tau (x_2^2 + x_3^2)\, d\tau \qquad \begin{pmatrix} 1\ 2\ 3 \\ 1\ 2 \ldots 6 \end{pmatrix} \tag{1.5}$$

$$\vartheta_{23} = Mx_{2c}x_{3c} - \sigma\rho_1 \int_0^l [(a+s)(u_{32} - u_{35} + u_{23} - u_{26}) + u_{21}u_{31} + u_{24}u_{34}]\, ds - \\ - \rho_2 \int_\tau x_2x_3 d\tau \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 \ldots & 6 \end{pmatrix}'$$

Here M is the system's mass, J_i are the principal central moments of inertia of the rigid body with undeformed rods, σ is the area of the rods' cross section, ρ_1 and ρ_2 are the densities of the rods and of the liquid, τ is the region of the space $Ox_1x_2x_3$ occupied at the current instant by the liquid, x_{ic} are the components of the radius-vector of the system's center of mass C relative to point O, to be computed by the formula

$$Mx_{1c} = \sigma\rho_1 \int_0^l (u_{12} + u_{15} + u_{13} + u_{16})\, ds + \rho_2 \int_\tau x_1 d\tau \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 \ldots & 6 \end{pmatrix} \tag{1.6}$$

The symbol $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 \ldots & 6 \end{pmatrix}$ on the right denotes that to obtain two other formulas we should make, in the expression presented, a cyclic permutation of (123) of the indices of the quantities x_i, γ_i, β_i, J_i, ϑ_{ij} and of the first indices of the constants I_{ij} and the functions u_{ij} together with their derivatives, and a cyclic permutation of (12...6) of the second indices of the constants I_{ij} and the functions u_{ij} and their derivatives; the primes over these symbols signify that when the second index of the functions u_{ij} and their derivatives changes from three to four and from six to one the sign in front of them should be replaced by the opposite one. In (1.5) and (1.6), x_{ic} is computed to within terms of the first order of smallness relative to the quantities u_{ij}, $u_{i,3+j}$ $(i, j = 1, 2, 3;$ $i \neq j)$, while ϑ_{ij} is computed to within terms of the second order relative to these same quantities and their derivatives.

For the kinetic energy T of the system in its motion relative to the orbital coordinate axes system, we have the expression

$$T = \frac{1}{2}\omega\cdot\Theta^c\cdot\omega + \omega \int_M (\mathbf{r} - \mathbf{r}_c)\times(\mathbf{r}^\cdot - \mathbf{r}_c^\cdot)\, dm + \frac{1}{2}\int_M (\mathbf{r}^\cdot - \mathbf{r}_c^\cdot)^2 dm \left(\mathbf{r}^\cdot = \frac{d\mathbf{r}}{dt}\right) \tag{1.7}$$

where $\omega = \omega_1\mathbf{i}_1 + \omega_2\mathbf{i}_2 + \omega_3\mathbf{i}_3$ is the body's relative angular velocity vector and dm is a mass element of the system. The mecnanical system being considered admits of a generalized integral of energy $T + W = \text{const}$, where W is the altered potential energy of the system

$$W = {}^1/_2\, \Omega^2\, [3\gamma\cdot\Theta^c\cdot\gamma - \beta\cdot\Theta^c\cdot\beta - \operatorname{sp}\Theta^c] + \Pi_d$$

Since γ_i, β_i are connected by relations (1.1), instead of W we shall consider below the functional

$$W_* = W - {}^1/_2\, \Omega^2\, (\pi_1\chi_1 + 2\pi_2\chi_2 + \pi_3\chi_3)$$

where π_1, π_2, π_3 are undetermined Lagrange multipliers.

2. Let us find the system's position of relative equilibrium. The equations of relative equilibrium, the natural boundary conditions, and the equation of the free surface S of the liquid at the relative equilibrium position are found from the principle of feasible displacements by computing and equating to zero the first variation δW_* of functional W_*. They have the form

$$(3\vartheta_{11} - \pi_1)\gamma_1 + 3(\vartheta_{12}\gamma_2 + \vartheta_{13}\gamma_3) - \pi_2\beta_1 = 0 \quad (1\ 2\ 3) \tag{2.1}$$
$$(\vartheta_{11} + \pi_3)\beta_1 + \vartheta_{12}\beta_2 + \vartheta_{13}\beta_3 + \pi_2\gamma_1 = 0 \quad (1\ 2\ 3)$$

$$(\beta_2^2 - 3\gamma_2^2)\ (u_{21} - x_{2c}) + (\beta_2\beta_3 - 3\gamma_2\gamma_3)\ (u_{31} - x_{3c}) + (\beta_1\beta_2 - 3\gamma_1\gamma_2)\ \times$$
$$\times\ (a + s - x_{1c}) + (\beta_1^2 - 3\gamma_1^2)\ \{[a(l - s) + {}^1/_2\ (l^2 - s^2)]u'_{21}\}' + E_* I_{31} \Omega^{-2} \times$$
$$\times\ u_{21}^{IV} = 0\ ({}_1{}^1{}_2{}^2 \ldots {}^3{}_6)$$
$$(\beta_3^2 - 3\gamma_3^2)(u_{31} - x_{3c}) + (\beta_2\beta_3 - 3\gamma_2\gamma_3)(u_{21} - x_{2c}) + (\beta_3\beta_1 - 3\gamma_3\gamma_1) \times$$
$$\times\ (a + s - x_{1c}) + (\beta_1^2 - 3\gamma_1^2)\{[a\ (l - s) + {}^1/_2\ (l^2 - s^2)]\ u_{31}'\}' + E_* I_{21} \times$$
$$\times\ \Omega^{-2}\ u_{31}^{IV} = 0\ ({}_1{}^1{}_2{}^2 \ldots {}_6{}^3) \qquad (2.2)$$
$$(\beta_2^2 - 3\gamma_2^2)\ (u_{24} - x_{2c}) + (\beta_2\beta_3 - 3\gamma_2\gamma_3)\ (u_{34} - x_{3c}) - (\beta_1\beta_2 - 3\gamma_1\gamma_2) \times$$
$$\times\ (a + s + x_{1c}) + (\beta_1^2 - 3\gamma_1^2)\ \{[a\ (l - s) + {}^1/_2\ (l^2 - s^2)]u_{24}'\}' + E_* I_{34} \times$$
$$\times\ \Omega^{-2} u_{24}^{IV} = 0\ ({}_1{}^1{}_2{}^2 \ldots {}^3{}_6)$$
$$(\beta_3^2 - 3\gamma_3^2)\ (u_{34} - x_{3c}) + (\beta_2\beta_3 - 3\gamma_2\gamma_3)\ (u_{24} - x_{2c}) - (\beta_3\beta_1 - 3\gamma_3\gamma_1) \times$$
$$\times\ (a + s + x_{1c}) + (\beta_1^2 - 3\gamma_1^2)\{[a\ (l - s) + {}^1/_2\ (l^2 - s^2)]u_{34}'\}' + E_* I_{24} \times$$
$$\times\ \Omega^{-2} u_{34}^{IV} = 0\ ({}_1{}^1{}_2{}^2 \ldots {}_6{}^3)\ (E = E_* \sigma\rho_1) \qquad (2.3)$$
$$u_{21}'' = u_{31}'' = u_{24}'' = u_{34}'' = 0,\ u_{21}''' = u_{31}''' = u_{24}''' = u_{34}''' = 0 \quad \begin{pmatrix} 1\ 2\ 3 \\ 1\ 2 \ldots 6 \end{pmatrix} \quad \text{for } s = l$$
$$U \equiv 3\,[(\mathbf{r} - \mathbf{r}_c)\cdot \boldsymbol{\gamma}]^2 - [(\mathbf{r} - \mathbf{r}_c)\cdot \boldsymbol{\beta}]^2 = c = \text{const} \qquad (2.4)$$

In Eq. (2.4) the value of the constant c is determined by the amount of liquid in the body's cavity. With respect to surface (2.4) the liquid is assumed to be on that side of it for which $U > c$. In the orbital coordinate axes system Eq. (2.4) of the liquid's free surface S is written in the form $U \equiv 3z^2 - y^2 = c$
and is the surface of a hyperbolic cylinder with a generator parallel to a tangent to the orbit.

3. Equations (2.1), (2.2) and boundary conditions (1.3), (2.3) admit of the solution

$$\gamma_1 = \gamma_2 = \beta_1 = \beta_3 = 0, \quad \gamma_3 = \beta_2 = 1, \quad u_{ij} = u_{i,\,3+j} \equiv 0 \quad (i,\ j = 1, 2,\ 3;\ i \neq j)$$
$$\pi_1 = 3\mathring{\vartheta}_{33}, \quad \pi_2 = 0,\ \pi_3 = -\vartheta_{22}^\circ \qquad (3.1)$$

The equation for the liquid's free surface S° has the form

$$U \equiv 3x_3^2 - x_2^2 = c^\circ \qquad (3.2)$$

This solution exists when the conditions $\mathring{\vartheta}_{23} = \mathring{\vartheta}_{31} = \mathring{\vartheta}_{12} = 0$, $\mathring{x}_{2c} = \mathring{x}_{3c} = 0$ are fulfilled. The superscript zero indicates that the corresponding quantity is computed for solution (3.1). Taking motion (3.1) as being unperturbed, we study its stability. For simplicity of computation we take it that $\mathring{x}_{1c} = 0$. We obtain the sufficient stability conditions from the theorem in [1] as the sufficient conditions for the positive definiteness of the second variation $\delta^2 W_*$ for the solution (3.1) in the metric with respect to which the functional W_* is continuous.

In the perturbed motion we set $\gamma_3 = 1 + \delta\gamma_3$, $\beta_2 = 1 + \delta\beta_2$, while we retain the previous notation for the remaining quantities. In a neighborhood of solution (3.1) relations (1.1) lead, in the first approximation, to the equalities $\delta\gamma_3 = \delta\beta_2 = 0$. Therefore, for computing $\delta^2 W_*$ we can formally take $\beta_2 = \gamma_3 = 1$; then for $\delta^2 W_*$ we obtain the expression

$$\delta^2 W_* = \Omega^2 [(\overset{\circ}{\vartheta}_{22} - \overset{\circ}{\vartheta}_{11})\beta_1^2 + 3(\overset{\circ}{\vartheta}_{11} - \overset{\circ}{\vartheta}_{33})\gamma_1^2 + 4(\overset{\circ}{\vartheta}_{22} - \overset{\circ}{\vartheta}_{33})\gamma_2^2] +$$
$$+ M\Omega^2(3x_{3c}^2 - x_{2c}^2) + 2\Pi_d + 2\Omega^2\sigma\rho_1 \int_0^l (a+s)[\beta_1(u_{21} - u_{24} + u_{12} - u_{15}) -$$
$$- 3\gamma_1(u_{13} - u_{16} + u_{31} - u_{34}) - 4\gamma_2(u_{32} - u_{35} + u_{23} - u_{26})]\, ds +$$
$$+ \Omega^2\sigma\rho_1 \int_0^l \Big\{u_{21}^2 + u_{24}^2 + u_{23}^2 + u_{26}^2 - 3(u_{31}^2 + u_{34}^2 + u_{32}^2 + u_{35}^2) +$$
$$+ \Big[a(l-s) + \frac{1}{2}(l^2 - s^2)\Big][3(u_{13}'^2 + u_{16}'^2 + u_{23}'^2 + u_{26}'^2) -$$
$$- (u_{12}'^2 + u_{15}'^2 + u_{32}'^2 + u_{35}'^2)]\Big\}\, ds + \Omega^2\Gamma_2 \qquad (3.3)$$

Here Γ_2 is the part of expression

$$\Gamma = -\rho_2 \int_{\Delta\tau} \sum_{(1\,2\,3)} [(3\gamma_1^2 - \beta_1^2)x_1^2 + 2(3\gamma_2\gamma_3 - \beta_2\beta_3)x_2x_3]\,|_{\beta_2=\gamma_3=1} d\tau \qquad (3.4)$$

which is quadratic in $\beta_1, \gamma_1, \gamma_2, u_{ij}$. The integral over the region $\Delta\tau$ should be understood as

$$\int_{\Delta\tau} \Phi\, d\tau = \int_{\tau} \Phi\, d\tau - \int_{\tau^\circ} \Phi\, d\tau$$

where τ^c and τ are the regions of space $Ox_1x_2x_3$ occupied by the liquid under unperturbed and perturbed motions of the system, bounded by the wetted part of the cavity's surface and, respectively, by the parts S° and S of surfaces (3.2) and (2.4) included inside the body's cavity; here in (2.4), $c = c^\circ + 2\Delta c$, and the value of the constant Δc is determined from the condition of eqaulity of the volumes of regions τ° and τ. We remark that surfaces S° and S may consist of several pieces.

The relations

$$Mx_{2c} = \sigma\rho_1 \int_0^l (u_{23} + u_{26} + u_{21} + u_{24})\, ds + \rho_2 \int_{\Delta\tau} x_2 d\tau$$
$$Mx_{3c} = \sigma\rho_1 \int_0^l (u_{31} + u_{34} + u_{32} + u_{35})\, ds + \rho_2 \int_{\Delta\tau} x_3 d\tau$$

together with the condition of equality of the volumes of regions τ° and τ form a system of three equations for determining the three quantities x_{2c}, x_{3c}, Δc as functions of β_1, γ_1, γ_2, u_{ij}. If the surface S° has three planes of symmetry and they are the coordinate planes x_2x_3, x_3x_1, x_1x_2, then the equations indicated simplify significantly and yield

$$x_{2c} = M_1 x_{2c}^{(1)}(M - M^{(+)})^{-1}, \quad x_{3c} = M_1 x_{3c}^{(1)}(M + M^{(-)})^{-1}, \quad \Delta c = 0 \qquad (3.5)$$

after which we obtain from (3.4),

$$\Gamma_2 = -J_{23}^{(+)}\beta_1^2 - 3J_{23}^{(-)}\gamma_1^2 - 16J_{12}^{(+)}\gamma_2^2 + M^{(+)}(M - M^{(+)})^{-2}(M_1 x_{2c}^{(1)})^2 +$$
$$+ 3M^{(-)}(M + M^{(-)})^{-2}(M_1 x_{3c}^{(1)})^2 \qquad (3.6)$$

$$M_1 x_{2c}^{(1)} = \sigma\rho_1 \int_0^l (u_{21} + u_{24} + u_{23} + u_{26})\, ds, \quad M_1 x_{3c}^{(1)} = \sigma\rho_1 \int_0^l (u_{31} + u_{34} + u_{32} + u_{35}) ds$$

$$M^{(+)} = \rho_2 \int_{S^\circ} \frac{x_2^2 dS}{\sqrt{9x_3^2 + x_2^2}} = \rho_2 \int_{S_{31}^\circ} |x_2|\, dx_3 dx_1 = \rho_2 \int_{\tau^{(+)}} d\tau$$

$$M^{(-)} = \rho_2 \int_{S^\circ} \frac{3x_3^2 dS}{\sqrt{9x_3^2 + x_2^2}} = \rho_2 \int_{S_{12}^\circ} |x_3|\, dx_1 dx_2 = \rho_2 \int_{\tau^{(-)}} d\tau$$

$$J_{12}^{(+)} = \rho_2 \int_{S^\circ} \frac{x_2^2 x_3^2 dS}{\sqrt{9x_3^2 + x_2^2}} = \rho_2 \int_{\tau^{(+)}} x_3^2 d\tau, \qquad J_{23}^{(+)} = \rho_2 \int_{S^\circ} \frac{x_1^2 x_2^2 dS}{\sqrt{9x_3^2 + x_2^2}} =$$

$$= \rho_2 \int_{\tau^{(+)}} x_1^2 d\tau$$

$$J_{23}^{(-)} = \rho_2 \int_{S^\circ} \frac{3x_3^2 x_1^2 dS}{\sqrt{9x_3^2 + x_2^2}} = \rho_2 \int_{\tau^{(-)}} x_1^2 d\tau$$

Here M_1 is the mass of the rods, $\overset{\circ}{S}_{12}$ and $\overset{\circ}{S}_{31}$ are the projections of surface S° onto the planes x_1x_2 and x_3x_1, while $\tau^{(+)}$ and $\tau^{(-)}$ are regions of space $Ox_1x_2x_3$, whose meanings are obvious. Figure 1 shows these regions for the case of a cylindrical cavity

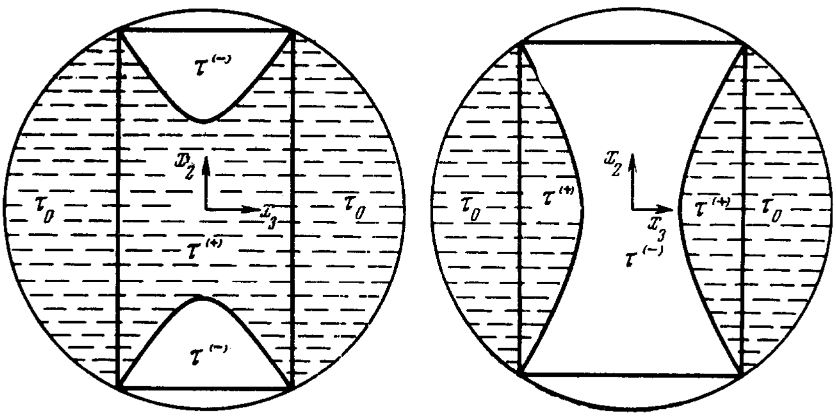

Fig. 1

with a generator parallel to the x_1-axis. The quantities $M^{(+)}$ and $M^{(-)}$ represent the masses which the liquid would have if it occupied the regions $\tau^{(+)}$ and $\tau^{(-)}$, respectively; $J_{23}^{(-)}$ is the moment of inertia relative to plane x_2x_3 of the liquid filling the region $\tau^{(-)}$, while $J_{12}^{(+)}$ and $J_{23}^{(+)}$ are the moments of inertia relative to planes x_1x_2 and x_2x_3 ot the liquid in region $\tau^{(+)}$. The quantities $M^{(+)}$, $M^{(-)}$, $J_{23}^{(-)}$, $J_{23}^{(+)}$, $J_{12}^{(+)}$, which arise due to the presence of the liquid's free surface, could be called the apparent additional masses and the apparent additional moments of inertia.

Substituting (3.5) and (3.6) into (3.3) we can represent $\delta^2 W_*$ as

$$\delta^2 W_* = \Omega^2 (\vartheta_{22}^\circ - \vartheta_{11}^\circ - J_{23}^{(+)})^{-1} \Big\{ (\vartheta_{22}^\circ - \vartheta_{11}^\circ - J_{23}^{(+)})\, \beta_1 +$$

$$+ \sigma\rho_1 \int_0^l (a + s)(u_{21} - u_{24} + u_{12} - u_{15})\, ds \Big\}^2 +$$

$$+ 3\Omega^2 (\vartheta_{11}^\circ - \vartheta_{33}^\circ - J_{23}^{(-)})^{-1} \Big\{ (\vartheta_{11}^\circ - \vartheta_{33}^\circ - J_{23}^{(-)})\, \gamma_1 -$$

$$- \sigma\rho_1 \int_0^l (a + s)(u_{13} - u_{16} + u_{31} - u_{34})\, ds \Big\}^2 +$$

$$+ 4\Omega^2 (\vartheta_{22}^\circ - \vartheta_{33}^\circ - 4J_{12}^{(+)})^{-1} \Big\{(\vartheta_{22}^\circ - \vartheta_{33}^\circ - 4J_{12}^{(+)}) \gamma_2 - \tag{3.7}$$

$$- \sigma\rho_1 \int_0^l (a+s)(u_{23} - u_{26} + u_{32} - u_{35})\, ds\Big\}^2 + 3\Omega^2 (M + M^{(-)})^{-1} (M_1 x_{3c}^{(1)})^2 + V$$

$$V(u) = 2\Pi_d - \Omega^2 (M - M^{(+)})^{-1} \Big\{\sigma\rho_1 \int_0^l (u_{21} + u_{24} + u_{23} + u_{26})\, ds\Big\}^2 -$$

$$- \Omega^2 (\vartheta_{22}^\circ - \vartheta_{11}^\circ - J_{23}^{(+)})^{-1} \Big\{\sigma\rho_1 \int_0^l (a+s)(u_{21} - u_{24} + u_{12} - u_{15})\, ds\Big\}^2 -$$

$$- 3\Omega^2 (\vartheta_{11}^\circ - \vartheta_{33}^\circ - J_{23}^{(-)})^{-1} \Big\{\sigma\rho_1 \int_0^l (a+s)(u_{13} - u_{16} + u_{31} - u_{34})\, ds\Big\}^2 -$$

$$- 4\Omega^2 (\vartheta_{22}^\circ - \vartheta_{33}^\circ - 4J_{12}^{(+)})^{-1} \Big\{\sigma\rho_1 \int_0^l (a+s)(u_{23} - u_{26} + u_{32} - u_{35})\, ds\Big\}^2 +$$

$$+ \Omega^2 \sigma\rho_1 \int_0^l \Big\{u_{21}^2 + u_{24}^2 + u_{23}^2 + u_{26}^2 - 3(u_{31}^2 + u_{34}^2 + u_{32}^2 + u_{35}^2) +$$

$$+ \Big[a(l-s) + \frac{1}{2}(l^2 - s^2)\Big] 3(u_{13}'^2 + u_{16}'^2 + u_{23}'^2 + u_{26}'^2) -$$

$$- (u_{12}'^2 + u_{15}'^2 + u_{32}'^2 + u_{35}'^2)]\Big\}\, ds \tag{3.8}$$

Let there be fulfilled the conditions

$$\vartheta_{22}^\circ - \vartheta_{11}^\circ - J_{23}^{(+)} > 0, \quad \vartheta_{11}^\circ - \vartheta_{33}^\circ - J_{23}^{(-)} > 0, \quad \vartheta_{22}^\circ - \vartheta_{33}^\circ - 4J_{12}^{(+)} > 0 \tag{3.9}$$

being the sufficient stability conditions for the relative equilibrium position (3.1) of a rigid body with undeformed ($u_{ij} \equiv 0$) rods and with a liquid in its cavity. Then, by using inequalities of the form

$$\Big\{\sigma\rho_1 \int_0^l w\, ds\Big\}^2 \leqslant m\sigma\rho_1 \int_0^l w^2 ds, \quad \Big\{\sigma\rho_1 \int_0^l (a+s) w\, ds\Big\}^2 \leqslant J\sigma\rho_1 \int_0^l w^2 ds$$

where $m = \sigma l \rho_1$ is the mass of one rod and J is its moment of inertia relative to point O, from (3.8) we obtain with due regard to (1.4), the inequality

$$V(u) \geqslant \sigma\rho_1 \int_0^l \{E_* (I_{31} u_{21}''^2 + I_{21} u_{31}''^2 + I_{12} u_{32}''^2 + I_{32} u_{12}''^2 + I_{23} u_{13}''^2 + I_{13} u_{23}''^2 + I_{31} u_{24}''^2 +$$

$$+ I_{24} u_{34}''^2 + I_{15} u_{35}''^2 + I_{35} u_{15}''^2 + I_{26} u_{16}''^2 + I_{16} u_{26}''^2) + \Omega^2 [a(l-s) + {}^1/_2 (l^2 - s^2)] \times$$

$$\times [3(u_{13}'^2 + u_{16}'^2 + u_{23}'^2 + u_{26}'^2) - (u_{12}'^2 + u_{15}'^2 + u_{32}'^2 + u_{35}'^2)] + \Omega^2 [u_{21}^2 + u_{24}^2 + u_{23}^2 +$$

$$+ u_{26}^2 - 3(u_{31}^2 + u_{34}^2 + u_{32}^2 + u_{35}^2)] - J\Omega^2 (\vartheta_{22}^\circ - \vartheta_{11}^\circ - J_{23}^{(+)})^{-1} (u_{21} - u_{24} +$$

$$+ u_{12} - u_{15})^2 - 3J\Omega^2 (\vartheta_{11}^\circ - \vartheta_{33}^\circ - J_{23}^{(-)})^{-1} (u_{13} - u_{16} + u_{31} - u_{34})^2 -$$

$$- 4J\Omega^2 (\vartheta_{22}^\circ - \vartheta_{33}^\circ - 4J_{12}^{(+)})^{-1} (u_{32} - u_{35} + u_{23} - u_{26})^2 -$$

$$- \Omega^2 m (M - M^{(+)})^{-1} (u_{21} + u_{24} + u_{23} + u_{26})^2\}\, ds \tag{3.10}$$

Let us consider the following variational problems. Find the minima $\nu_1, \nu_2, \ldots, \nu_6$ of the functionals

$$\Phi_\alpha(u_{2\alpha}, u_{3\alpha}) = \left\{\int_0^l [u_{2\alpha}^2 + u_{3\alpha}^2 + \sigma(u_{2\alpha}'^2 + u_{3\alpha}'^2)]\,ds\right\}^{-1} \int_0^l E_*(I_{3\alpha}u_{2\alpha}''^2 + I_{2\alpha}u_{3\alpha}''^2)\,ds$$

$$(\alpha = 1,4)$$

$$\Phi_\beta(u_{3\beta}, u_{1\beta}) = \left\{\int_0^l [u_{3\beta}^2 + u_{1\beta}^2 + \sigma(u_{3\beta}'^2 + u_{1\beta}'^2)]\,ds\right\}^{-1} \int_0^l \Big\{E_*(I_{1\beta}u_{3\beta}''^2 + I_{3\beta}u_{1\beta}''^2) -$$

$$- \Omega^2\left[a(l-s) + \frac{1}{2}(l^2 - s^2)\right](u_{3\beta}'^2 + u_{1\beta}'^2)\Big\}\,ds \quad (\beta = 2,5)$$

$$\Phi_\gamma(u_{1\gamma}, u_{2\gamma}) = \left\{\int_0^l [u_{1\gamma}^2 + u_{2\gamma}^2 + \sigma(u_{1\gamma}'^2 + u_{2\gamma}'^2)]\,ds\right\}^{-1} \times \tag{3.11}$$

$$\times \int_0^l \Big\{E_*(I_{3\gamma}u_{1\gamma}''^2 + I_{1\gamma}u_{3\gamma}''^2) + 3\Omega^2\left[a(l-s) + \frac{1}{2}(l^2 - s^2)\right](u_{1\gamma}'^2 + u_{2\gamma}'^2)\Big\}\,ds$$

$$(\gamma = 3,6)$$

in the class of functions $u_{ij}(s)$ $(0 \leqslant s \leqslant l)$ continuously differentiable upto fourth order, satisfying conditions (1.3). The constants ν_j can be computed also as smallest eigenvalues of the corresponding boundary value problems, independent one from the other, for the functions u_{ij}.

From (3.10) and (3.11) we obtain the inequality

$$V(u) \geqslant \sigma\rho_1 \int_0 \{\nu_1[u_{21}^2 + u_{31}^2 + \sigma(u_{21}'^2 + u_{31}'^2)] + \nu_2[u_{32}^2 + u_{12}^2 +$$

$$+ \sigma(u_{32}'^2 + u_{12}'^2)] + \nu_3[u_{13}^2 + u_{23}^2 + \sigma(u_{13}'^2 + u_{23}'^2)] + \nu_4[u_{24}^2 + u_{34}^2 +$$

$$+ \sigma(u_{24}'^2 + u_{34}'^2)] + \nu_5[u_{35}^2 + u_{15}^2 + \sigma(u_{35}'^2 + u_{15}'^2)] + \nu_6[u_{16}^2 + u_{26}^2 +$$

$$+ \sigma(u_{16}'^2 + u_{26}'^2)] + \Omega^2[u_{21}^2 + u_{24}^2 + u_{23}^2 + u_{26}^2 - 3(u_{31}^2 + u_{34}^2 + u_{32}^2 +$$

$$+ u_{35}^2)] - J(\vartheta_{22}^\circ - \vartheta_{11}^\circ - J_{23}^{(+)})^{-1}\Omega^2(u_{21} - u_{24} + u_{12} - u_{15})^2 -$$

$$- 3J(\vartheta_{11}^\circ - \vartheta_{33}^\circ - J_{23}^{(-)})^{-1}\Omega^2(u_{13} - u_{16} + u_{31} - u_{34})^2 - 4J(\vartheta_{22}^\circ - \vartheta_{33}^\circ -$$

$$- 4J_{12}^{(+)})^{-1}\Omega^2(u_{32} - u_{35} + u_{23} - u_{26})^2 -$$

$$- m(M - M^{(+)})^{-1}\Omega^2(u_{21} + u_{24} + u_{23} + u_{26})^2\}\,ds \tag{3.12}$$

For simplicity of computation we take

$$I_{21} = I_{31} = I_{24} = I_{34} = I_1, \quad I_{32} = I_{12} = I_{35} = I_{15} = I_2$$

$$I_{13} = I_{23} = I_{16} = I_{26} = I_3$$

Then $\nu_1 = \nu_4$, $\nu_2 =$ $\nu_3 = \nu_6$, and the Sylvester conditions for the positive definiteness of a quadratic form in the quantities u_{ij}, u_{ij} occurring under the integral sign in (3.12), reduce to the inequalities: $3\Omega^2 < \nu_1$ (ν_1 is independent of Ω)

$$\vartheta_{22}^\circ - \vartheta_{11}^\circ - J_{23}^{(+)} > \frac{2J\Omega^2}{\nu_1 + \Omega^2}\max\left\{1;\ \frac{\nu_1 + \nu_2 + \Omega^2}{\nu_2}\right\} > 0 \tag{3.13}$$

$$\vartheta_{11}^\circ - \vartheta_{33}^\circ - J_{23}^{(-)} > \frac{6J\Omega^2}{\nu_3}\max\left\{1;\ \frac{\nu_3 + \nu_1 - 3\Omega^2}{\nu_1 - 3\Omega^2}\right\} > 0$$

$$\vartheta_{22}^\circ - \vartheta_{33}^\circ - 4J_{12}^{(+)} > \frac{8J\Omega^2}{\nu_3 + \Omega^2}\max\left\{1;\ \frac{\nu_2 + \nu_3 - 2\Omega^2}{\nu_2 - 3\Omega^2}\right\} > 0$$

$$\frac{2m}{M - M^{(+)}} < \frac{\nu_3 + \Omega^2}{\Omega^2}\min\left\{1;\ \frac{\nu_1 + \Omega^2}{\nu_3 + \nu_1 + 2\Omega^2}\right\}$$

4. Along with the relative equilibrium position (3.1) of the system, for which the constant c° in Eq. (3.2) for the liquid's free surface S° is taken to be nonzero (otherwise the liquid's continuity is violated in the system's perturbed motion),we consider at the instant $t > t_0$ the system's position corresponding to some perturbed motion of the system. We introduce into consideration the distance h of the perturbed free surface S of the liquid from the unperturbed S°, and also the deviation ∇ of the liquid's form τ, corresponding to the perturbed state of the system, from the equilibrium form τ° [4].

Let us consider the set whose elements are the quantities $\beta_1, \gamma_1, \gamma_2, h$ and the functions

$$u_{ij}(s,t),\ 0 \leqslant s \leqslant l,\ t \geqslant t_0\ (i = 1,2,3;\ j = 1,\ 2,\ \ldots,\ 6;\ j \neq i,\ 3+i).$$

Regarding the functions u_{ij} we assume that they satisfy conditions (1.3) as well as specific smoothness conditions (it suffices to require the continuity of the functions u_{ij}, u_{ij}', $\ldots$, u_{ij}^{IV}, $\dot{u}_{ij}$, $\ddot{u}_{ij}$, $\dot{u}_{ij}''$, $\ddot{u}_{ij}''$). We also take it that for a specified distance h the magnitude of the corresponding deviation ∇ satisfies the condition $\nabla > \varepsilon_0 h$, where ε_0 is some fixed sufficiently-small positive number [4]. In this set we introduce two metrics

$$Q_0(\beta,\gamma,h,u) = L^3(\beta_1^2+\gamma_1^2+\gamma_2^2) + Lh^2 + \sum_{i,j}\int_0^l (u_{ij}^2 + l^2 u_{ij}'^2 + l^4 u_{ij}''^2)\,ds$$

$$Q(\beta,\gamma,h,u) = L^3(\beta_1^2+\gamma_1^2+\gamma_2^2) + Lh^2 + \sum_{i,j}\int_0^l (u_{ij}^2 + l^2 u_{ij}'^2)\,ds$$

In the neighborhood of the unperturbed motion (3.1) metric Q and functional W_* are continuous with respect to metric Q_0, i.e., for any $\varepsilon > 0$ we can find $\delta(\varepsilon) > 0$ such that when the condition $Q_0(\beta,\ \gamma,\ h,\ u) < \delta$ is fulfilled there hold the inequalities $Q < \varepsilon$, $|\,W_* - W_*^\circ\,| < \varepsilon$, where W_*° is the value of W_* for solution (3.1). The functionals Q_0 and Q characterize the deviation of the perturbed state of the system from the relative equilibrium position (3.1). The deviation of the velocities of the points of the system in perturbed motion from their zero values in relative equilibrium (3.1) is characterized by the magnitude T of kinetic energy (1.7) of relative motion of the system as well as by the functional

$$P(\omega, u^\cdot, x^\cdot) = \sum_i J_i\omega_i^2 + \sum_{i,j}\sigma\rho_1\int_0^l \dot{u}_{ij}^2\,ds + \rho_2\int_\tau (\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)\,d\tau$$

For any admissible values of the quantities β_1, γ_1, γ_2, h, u_{ij} satisfying the condition $Q_0(\beta,\ \gamma,\ h,\ u) < N$, where N is a fixed sufficiently-small positive constant, $T \to 0$ as $P \to 0$.

From (3.7) and (3.12) it follows that when conditions (3.13) are fulfilled $\delta^2 W_*$ is a functional which is positive definite in the metric Q. Because W_* is continuous in metric Q_0 it follows that the functional W_* is positive definite in the metric Q. On the basis of a theorem in [1] we conclude that inequalities (3.13) are sufficient stability conditions for the relative equilibrium (3.1) with respect to the functionals Q_0, Q, T and P. This signifies that for every arbitrarily small positive numbers L_1 and L_2 we may choose positive numbers N_1 and N_2 such that under every admissible values of the magnitudes of the direction cosines β_i, γ_i, the separation h, the deviation ∇ $(\nabla > \varepsilon_0 h)$, the elastic displacements u_{ij}, the relative angular velocities ω of the body, the velocities $\dot{u}_{ij}$ of the elastic rods, and the velocities $\dot{x}_i$ of the liquid, satisfying the conditions

$$Q_0(\beta, \gamma, h, u)|_{t=t_0} < N_1, \qquad P(\omega, u^\cdot, x^\cdot)|_{t=t_0} < N_2 \tag{4.1}$$

for $t \geqslant t_0$ or at least until $\nabla > \varepsilon_0 h$ there are satisfied the inequalities

$$Q(\beta, \gamma, h, u) < L_1, \qquad T < L_2 \tag{4.2}$$

It follows, in particular, from (1.3), (4.1) and (4.2) that for sufficiently small values of N_1 and N_2 there ensues, for $t \geqslant t_0$, not only the first of inequalities (4.2) but also the inequalities $l^2 u_{ij}^2 < L_1$.

Conditions (3.13) show that the attachment of elastic rods to the body, just as the presence in the body's cavity of a liquid with a free surface, proves to have a destabilizing influence on the relative equilibrium of the undeformed system with a liquid "frozen" in the position of relative equilibrium (3.1). The first of conditions (3.13) imposes a specific upper bound on the magnitude Ω of the orbital angular velocity and is connected with the existence of the forms of the loss of stability of the rods.

As $E \to \infty$ the constants $\nu_1, \nu_2, \nu_3 \to \infty$ and conditions (3.13) turn into conditions (3.9) for the stability of relative equilibrium (3.1) in the circular orbit of a rigid body with nondeformable rods and with liquid in its cavity; here, if the liquid is absent or wholly fills the cavity, then $J_{12}^{(+)} = J_{23}^{(+)} = J_{23}^{(-)} = 0$. In case the liquid is absent conditions (3.9) turn into the well-known conditions $J_2 > J_1 > J_3$ for the stability of the relative equilibrium (3.1) of a rigid satellite in a circular orbit. We remark that also in the case of a viscous liquid inequalities (3.13) are sufficient conditions for the stability of the system's relative equilibrium being investigated.

5. Let us indicate the sufficient stability conditions for relative equilibrium (3.1) of a rigid body with a liquid and with one or two pairs of elastic rods.

1. A pair of rods in the position of relative equilibrium (3.1) is directed along the normal to the orbital plane

$$\vartheta_{22}^\circ - \vartheta_{11}^\circ - J_{23}^{(+)} > \frac{2J\Omega^2}{\nu_2}, \quad \vartheta_{11}^\circ - \vartheta_{33}^\circ - J_{23}^{(-)} > 0, \quad \vartheta_{22}^\circ - \vartheta_{33}^\circ - 4J_{12}^{(+)} > \frac{8J\Omega^2}{\nu_2 - 3\Omega^2} > 0 \tag{5.1}$$

2. A pair of rods in the position of relative equilibrium (3.1) is directed along a tangent to the orbit

$$\vartheta_{22}^\circ - \vartheta_{11}^\circ - J_{23}^{(+)} > \frac{2J\Omega^2}{\nu_1 + \Omega^2}, \quad \vartheta_{11}^\circ - \vartheta_{33}^\circ - J_{23}^{(-)} > \frac{6J\Omega^2}{\nu_1 - 3\Omega^2} > 0, \quad \vartheta_{22}^\circ - \vartheta_{33}^\circ - 4J_{12}^{(+)} > 0 \tag{5.2}$$

3. A pair of rods in the position of relative equilibrium (3.1) is directed along the normal to the orbit

$$\vartheta_{22}^\circ - \vartheta_{11}^\circ - J_{23}^{(+)} > 0, \quad \vartheta_{11}^\circ - \vartheta_{33}^\circ - J_{23}^{(-)} > \frac{6J\Omega^2}{\nu_3} > 0, \quad \vartheta_{22}^\circ - \vartheta_{33}^\circ - 4J_{12}^{(+)} > \frac{8J\Omega^2}{\nu_3 + \Omega^2} \tag{5.3}$$

4. Two pairs of rods in the position of relative equilibrium (3.1) are directed along the tangent and the normal to the orbit

$$\vartheta_{22}^\circ - \vartheta_{11}^\circ - J_{23}^{(+)} > \frac{2J\Omega^2}{\nu_1 + \Omega^2}, \quad \vartheta_{11}^\circ - \vartheta_{33}^\circ - J_{23}^{(-)} > \frac{6J\Omega^2}{\nu_3} \max\left\{1; \frac{\nu_1 + \nu_3 - 3\Omega^2}{\nu_1 - 3\Omega^2}\right\}$$

$$\vartheta_{22}^\circ - \vartheta_{33}^\circ - 4J_{12}^{(+)} > \frac{8J\Omega^2}{\nu_3 + \Omega^2}, \quad 3\Omega^2 < \nu_1, \quad \frac{2m}{M - M^{(+)}} < \frac{(\nu_3 + \Omega^2)(\nu_1 + \Omega^2)}{\Omega^2(\nu_3 + \nu_1 + 2\Omega^2)} \tag{5.4}$$

5. Two pairs of rods in the position of relative equilibrium (3.1) are directed along the tangent and the binormal to the orbit

$$\vartheta_{22}^\circ - \vartheta_{11}^\circ - J_{23}^{(+)} < \frac{2J\Omega^2}{\nu_1 + \Omega^2} \max\left\{1; \frac{\nu_1 + \nu_2 + \Omega^2}{\nu_2}\right\} > 0 \qquad (5.5)$$

$$\vartheta_{11}^\circ - \vartheta_{33}^\circ - J_{23}^{(-)} > \frac{6J\Omega^2}{\nu_1 - 3\Omega^2} > 0, \qquad \vartheta_{22}^\circ - \vartheta_{33}^\circ - 4J_{12}^{(+)} > \frac{8J\Omega^2}{\nu_2 - 3\Omega^2} > 0$$

6. Twc pairs of rods in the position of relative equilibrium (3.1) are directed along the normal and the binormal to the orbit

$$\vartheta_{22}^\circ - \vartheta_{11}^\circ - J_{23}^{(+)} > \frac{2J\Omega^2}{\nu_2} > 0, \qquad \vartheta_{11}^\circ - \vartheta_{33}^\circ - J_{23}^{(-)} > \frac{6\Omega^2}{\nu_3} > 0 \qquad (5.6)$$

$$\vartheta_{22}^\circ - \vartheta_{33}^\circ - 4J_{12}^{(+)} > \frac{8J\Omega^2}{\nu_3 + \Omega^2} \max\left\{1; \frac{\nu_3 + \nu_2 - 2\Omega^2}{\nu_2 - 3\Omega^2}\right\} > 0$$

Conditions (5.1) - (5.6) may be obtained from (3.13) as

$$\nu_1, \nu_3 \to \infty;\ \nu_2, \nu_3 \to \infty,\ \nu_1, \nu_2 \to \infty;\ \nu_2 \to \infty;\ \nu_3 \to \infty;\ \nu_1 \to \infty$$

respectively. Physically the passage to the limit $\nu_i \to \infty$ means that the rods directed along the x_i -axis are progressively "frozen" and in the limit they should be considered as being undeformed.

6. Meirovitch [2] has investigated the stability of one of the relative equilibrium positions on a circular orbit in a central Newtonian force field of a rigid body with two thin rectilinear elastic rods. Let us compare the results obtained above with the results in [2] in which the rod deformations are described, in the notation used above, by the vector-valued functions

$$\mathbf{u}_2(s, t) = u_{12}\mathbf{i}_1 + \mathbf{u}_{32}\mathbf{i}_3,\ \mathbf{u}_5(s, t) = u_{15}\mathbf{i}_1 + u_{35}\mathbf{i}_3 \quad (0 \leqslant s \leqslant l)$$

so that the axial components u_{22} and u_{25} of the elastic variables can be neglected. In the system's relative equilibrium position investigated the pair of rods are in the direction of the normal to the orbital plane, and the sufficient stability conditions have the form

$$C > B > A > 2m(a + l)^2$$

$$\Lambda = \left(3 + \frac{8m(a + l)^2}{C - A}\right)\Omega^2, \qquad \Lambda > \frac{2m(a + l)^2}{C - B}\Omega^2 \qquad (6.1)$$

Here Λ is the smallest eigenvalue of the boundary value problem

$$EIu^{\mathrm{IV}} = \sigma\rho_1 \Lambda u, \qquad u(0) = u'(0) = u''(l) = u'''(l) = 0$$

while the quantities A, B, C, by the author's assertion, are the principal central moments of inertia of the system in its undeformed state relative to the x_3, x_1, x_2 axes, respectively. Here we have not indicated the parameters with respect to which we have investigated the stability of the unperturbed motion being considered. Using the inequality

$$J = \sigma\rho_1 \int_0^l (a + s)^2\, ds < m(a + l)^2$$

condition (6.1) can be brought to the form

$$C > B > A > 2J, \qquad \Lambda > \left(3 + \frac{8J}{C - A}\right)\Omega^2, \qquad \Lambda > \frac{2J\Omega^2}{C - B} \qquad (6.2)$$

The sufficient stability conditions (5.1), obtained in Sect. 5, for the relative equilibrium position being considered here in the absence of a liquid can be represented in the form

$$J_2 > J_1 > J_3 > 0, \qquad \nu_2 > \left(3 + \frac{8J}{J_2 - J_3}\right)\Omega^2, \qquad \nu_2 > \frac{2J\Omega^2}{J_2 - J_1} \tag{6.3}$$

The resemblance of conditions (6.2) and (6.3) is a formal one since the quantities A, B, C are not constants, in spite of Meirovitch's assertion, but represent the functionals

$$A = J_3 + M_T x_{1c}^2, \qquad B = J_1 + M_T x_{3c}^2, \qquad C = J_2 + M_T (x_{1c}^2 + x_{3c}^2)$$

$$M x_{1c} = \sigma\rho_1 \int_0^l (u_{12} + u_{15})\, ds, \qquad M x_{3c} = \sigma\rho_1 \int_0^l (u_{32} + u_{35})\, ds$$

where M_T is the mass of the rigid body.

7. We now investigate the case when no external forces whatsoever act on the mechanical system under consideration, so that its center of mass moves rectilinearly and uniformly. We now assume that the coordinate axes system $Cxyz$ is König's system. The mechanical system being considered admits of the energy integral $T + \Pi_d = \text{const}$, where T is the system's kinetic energy in its motion relative to the König axes. We introduce into consideration a coordinate axes system $Czz'z''$ rotating around the z-axis with some angular velocity Ω. For the (xy)-plane there holds the area integral $\mathbf{G}\cdot\boldsymbol{\gamma} = k = \text{const}$, where $\mathbf{G}$ is the kinetic moment vector of the system relative to its center of mass as the system moves relative to the König axes, and $\boldsymbol{\gamma}$ is the unit vector on the z-axis. Denoting by $\mathbf{G}_*$ the system's kinetic moment vector relative to point C in its motion relative to the axes $Czz'z''$, we can represent the area integral as

$$\mathbf{G}_*\cdot\boldsymbol{\gamma} + J_*\Omega = k$$

where $J_* = \boldsymbol{\gamma}\cdot\boldsymbol{\Theta}^c\cdot\boldsymbol{\gamma}$ is the system's moment of inertia relative to the z-axis. We choose the quantity Ω such that the equality $\mathbf{G}_*\cdot\boldsymbol{\gamma} = 0$ holds at any instant. Then we have $J_*\Omega = k$, and the energy integral can be represented in the form $T_* + W = \text{const}$, where T_* is the kinetic energy of the system's relative motion and W is the changed potential energy of the system

$$W = \frac{k^2}{2J_*} + \Pi_d$$

Below, instead of W we shall consider the functional

$$W_* = W + {}^1/_2\lambda\,(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)$$

where λ is an undetermined Lagrange multiplier.

From the equality $\delta W_* = 0$ we obtain the equations of steady-state motion of the system, the natural boundary conditions (2.3) at the free ends of the rods, and the equation of the free surface of the liquid in steady-state motion. These equations have the form

$$(\vartheta_{11}\gamma_1 + \vartheta_{12}\gamma_2 + \vartheta_{13}\gamma_3)\Omega_0^2 - \lambda\gamma_1 = 0 \quad (1\ 2\ 3) \tag{7.1}$$

$$x_{3c} - (x_{1c}\gamma_1 + x_{2c}\gamma_2 + x_{3c}\gamma_3)\gamma_3 + (\gamma_3^2 - 1)u_{31} + (a + s)(u_{21}\gamma_2\gamma_3 + \gamma_3\gamma_1) +$$
$$+ (\gamma_1^2 - 1)\ [a(l - s) + {}^1/_2\,(l^2 - s^2)]u_{31}'\}' + E_* I_{21}\Omega_0^{-2} u_{31}^{\text{IV}} = 0 \qquad \begin{pmatrix} 1\,2\,3 \\ 1\,2\,\ldots\,6 \end{pmatrix}$$

$$x_{2c} - (x_{1c}\gamma_1 + x_{2c}\gamma_2 + x_{3c}\gamma_3)\gamma_2 + (\gamma_2^2 - 1)u_{21} + (a + s)(u_{31}\gamma_2\gamma_3 + \gamma_1\gamma_2) +$$
$$+ (\gamma_1^2 - 1)\ \{[a\,(l - s) + {}^1/_2\,(l - s^2)]u_{21}'\}' + E_* I_{31}\Omega_0^{-2} u_{21}^{\text{IV}} = 0 \qquad \begin{pmatrix} 1\,2\,3 \\ 1\,2\,\ldots\,6 \end{pmatrix}$$

$$x_{3c} - (x_{1c}\gamma_1 + x_{2c}\gamma_2 + x_{3c}\gamma_3)\gamma_3 + (\gamma_3^2 - 1)\, u_{34} + (a + s)\,(u_{24}\gamma_2\gamma_3 - \gamma_3\gamma_1) +$$
$$+ (\gamma_1^2 - 1)\,\{[a\,(l - s) + {}^1/_2\,(l^2 - s^2)]u_{34}'\}' + E_* I_{24}\Omega_0^{-2}u_{34}^{\mathrm{IV}} = 0 \quad \begin{pmatrix} 1\,2\,3 \\ 1\,2\cdots 6 \end{pmatrix}$$
$$x_{2c} - (x_{1c}\gamma_1 + x_{2c}\gamma_2 + x_{3c}\gamma_3)\gamma_2 + (\gamma_2^2 - 1)u_{24} + (a + s)\;(u_{34}\gamma_2\gamma_3 - \gamma_1\gamma_2) +$$
$$+ (\gamma_1^2 - 1)\,\{[a\,(l - s) + {}^1/_2(l^2 - s^2)]u_{24}'\}' + E_* I_{34}\Omega_0^{-2}u_{24}^{\mathrm{IV}} = 0 \quad \begin{pmatrix} 1\,2\,3 \\ 1\,2\cdots 6 \end{pmatrix}$$
$$U \equiv (\mathbf{r} - \mathbf{r}_c)^2 - [(\mathbf{r} - \mathbf{r}_c)\cdot\boldsymbol{\gamma}]^2 = c \tag{7.2}$$

Here Ω_0 is the magnitude of the angular velocity of the system's uniform rotation as one rigid body in steady-state motion, c is a constant determined by the amount of liquid in the body's cavity. With respect to the surface $U = c$ the liquid is located to that side of it for which $U > c$. In the König axes Eq. (7.2) has the form $U = x^2 + y^2 = c$.

8. Equations (7.1) and boundary conditions (1.3), (2.3) admit of the solution

$$\gamma_1 = \gamma_2 = 0,\ \gamma_3 = 1,\ \vartheta_{13}^\circ = \vartheta_{23}^\circ = 0,\ x_{1c}^\circ = x_{2c}^\circ = 0,\ \ \lambda = \vartheta_{33}^\circ\Omega_0^2 \tag{8.1}$$
$$u_{ij} = u_{i,3+j} \equiv 0 \qquad (i, j = 1, 2, 3;\ \ i \neq j)$$

This solution describes the uniform rotation of the system as one rigid body around the z-axis with an arbitrary angular velocity Ω_0; here the rods are found in the undeformed state, and the equation of the liquid's free surface has the form $U^\circ = x_1^2 + x_2^2 = c_0$

Let us investigate the stability of motion (8.1). For simplicity of computation we assume $x_{3c}^\circ = 0$. The sufficient stability conditions for the unperturbed motion (8.1) are obtained as conditions for the positive definiteness of the second variation $\delta^2 W_*$ of functional W_* for solution (8.1) under the condition that $\gamma_3 \equiv 1$. For $\delta^2 W_*$ there holds the expression

$$\delta^2 W_* = \Omega_0^2\,[(\vartheta_{33}^\circ - \vartheta_{11}^\circ)\,\gamma_1^2 - 2\vartheta_{12}^\circ\gamma_1\gamma_2 + (\vartheta_{33}^\circ - \vartheta_{22}^\circ)\,\gamma_2^2] + M\Omega_0^2\,(x_{1c}^2 +$$
$$+ x_{2c}^2) - \Omega_0^2\sigma\rho_1 \int_0^l \{u_{13}^2 + u_{16}^2 + u_{12}^2 + u_{15}^2 + u_{21}^2 + u_{24}^2 + u_{23}^2 + u_{26}^2 -$$
$$- 2\,(a + s)\,[\gamma_1\,(u_{13} - u_{16} + u_{31} - u_{34}) + \gamma_2\,(u_{23} - u_{26} + u_{32} - u_{35})] -$$
$$- [a\,(l - s) + {}^1/_2\,(l^2 - s^2)]\;(u_{12}'^2 + u_{15}'^2 + u_{21}'^2 + u_{24}'^2 + u_{31}'^2 + u_{34}'^2 +$$
$$+ u_{32}'^2 + u_{35}'^2)\}\, ds + 2\Pi_d + \Omega_0^2\Gamma_2 \tag{8.2}$$

Here Γ_2 is the quadratic part, relative to γ_1, γ_2, u_{ij} of the expression

$$\Gamma = -\rho_2 \int_{\Delta\tau} [x_1^2 + x_2^2 + x_3^2 - (x_1\gamma_1 + x_2\gamma_2 + x_3\gamma_3)^2]\,|_{\gamma_3=1} d\tau \tag{8.3}$$

In the case when the liquid's free surface S° possesses three planes of symmetry and they are the coordinate planes, for Γ_2 there holds the expression

$$\Gamma_2 = M^{(1)}x_{1c}^2 + M^{(2)}x_{2c}^2 - J_{12}^{(1)}\gamma_1^2 - J_{12}^{(2)}\gamma_2^2 \tag{8.4}$$
$$M^{(1)} = \rho_2 \int_{S^\circ} \frac{x_1^2 dS}{\sqrt{x_1^2 + x_2^2}} = \rho_2 \int_{\tau^{(1)}} d\tau, \qquad M^{(2)} = \rho_2 \int_{S^\circ} \frac{x_2^2 dS}{\sqrt{x_1^2 + x_2^2}} = \rho_2 \int_{\tau^{(2)}} d\tau$$
$$J_{12}^{(1)} = \rho_2 \int_{S^\circ} \frac{x_1^2 x_3^2 dS}{\sqrt{x_1^2 + x_2^2}} = \rho_2 \int_{\tau^{(1)}} x_3^2 d\tau, \qquad J_{12}^{(2)} = \rho_2 \int_{S^\circ} \frac{x_2^2 x_3^2 dS}{\sqrt{x_1^2 + x_3^2}} = \rho_2 \int_{\tau^{(2)}} x_3^2 d\tau$$

Here $M^{(1)}$, $M^{(2)}$, $J_{12}^{(1)}$, $J_{12}^{(2)}$ are the masses and the moments of inertia relative to the ($x_1 x_2$)-plane which the liquid would have if it filled the regions $\tau^{(1)}$ and $\tau^{(2)}$ of the ($x_1 x_2 x_3$)-space, whose geometric meaning is analogous to that of the regions $\tau^{(-)}$ and

$\tau^{(+)}$ considered earlier. The quantities $M^{(1)}$, $M^{(2)}$ and $J_{12}^{(1)}$, $J_{12}^{(2)}$, caused by the presence of a free surface of the liquid, could be called the apparent additional masses and moments of inertia of the system.

Suppose that the conditions

$$D_1 \equiv \vartheta_{33}^\circ - \vartheta_{11}^\circ - J_{12}^{(1)} > 0, \quad D_2 \equiv \vartheta_{33}^\circ - \vartheta_{22}^\circ - J_{12}^{(2)} > 0$$
$$D = D_1 D_2 - \vartheta_{12}^{\circ 2} > 0 \tag{8.5}$$

are fulfilled. Then, using the Schwarz inequality, from (8.2) we obtain, with due regard to (8.4), the inequality

$$\delta^2 W_* \geq \Omega_0^2 D_1^{-1} \Big\{ D_1 \gamma_1 - \vartheta_{12}^\circ \gamma_2 + \sigma\rho_1 \int_0^l (a+s)(u_{13} - u_{16} + u_{31} - u_{34})\, ds \Big\}^2 +$$
$$+ \Omega_0^2 D^{-1} D_1^{-1} \Big\{ D\gamma_2 + \sigma\rho_1 \int_0^l (a+s) [\vartheta_{12}^\circ (u_{13} - u_{16} + u_{31} - u_{34}) + D_1 (u_{23} -$$
$$- u_{26} + u_{32} - u_{35})] ds \Big\}^2 + \Omega_0^2 [(M + M^{(1)}) x_{1c}^2 + (M + M^{(2)}) x_{2c}^2] + \sigma\rho_1 V(u) \tag{8.6}$$

$$V(u) = \int_0^l \{ E_* (I_{23} u_{13}''^2 + I_{26} u_{16}''^2 + I_{31} u_{21}''^2 + I_{34} u_{24}''^2 + I_{12} u_{32}''^2 + I_{15} u_{35}''^2 +$$
$$+ I_{13} u_{23}''^2 + I_{16} u_{26}''^2 + I_{21} u_{31}''^2 + I_{24} u_{34}''^2 + I_{32} u_{12}''^2 + I_{35} u_{15}''^2) +$$
$$+ \Omega_0^2 [a(l-s) + {}^1/_2 (l^2 - s^2)] (u_{12}'^2 + u_{15}'^2 + u_{21}'^2 + u_{24}'^2 + u_{31}'^2 + u_{34}'^2 +$$
$$+ u_{32}'^2 + u_{35}'^2) - \Omega_0^2 (u_{12}^2 + u_{15}^2 + u_{13}^2 + u_{16}^2 + u_{21}^2 + u_{24}^2 + u_{23}^2 + u_{26}^2) -$$
$$- J D^{-1} \Omega_0^2 [D_1 (u_{23} - u_{26} + u_{32} - u_{35})^2 + D_2 (u_{13} - u_{16} + u_{31} - u_{34})^2 +$$
$$+ 2\vartheta_{12}^\circ (u_{23} - u_{26} + u_{32} - u_{35})(u_{13} - u_{16} + u_{31} - u_{34})] \} ds \tag{8.7}$$

For simplicity of computation we assume that

$$I_{13} = I_{16} = I_{23} = I_{26} = I_0, \; I_{12} = I_{15} = I_{21} = I_{24} = I_{31} = I_{34} =$$
$$= I_{32} = I_{35} = I_1$$

We consider the following variational problems. Find the minima $\varkappa_0$, $\varkappa_1$ of the functionals

$$\Phi_\alpha(u) = \Big\{ \int_0^l (u^2 + \sigma u'^2)\, ds \Big\}^{-1} \int_0^l \Big\{ E_* I_\alpha u''^2 - \alpha\Omega_0^2 \Big[a(l-s) + \frac{1}{2}(l^2 - s^2) \Big] u'^2 \Big\} ds$$
$$(\alpha = 0, 1) \tag{8.8}$$

in the class of functions $u(s)$, $0 \leqslant s \leqslant l$, continuously differentiable upto the fourth order, satisfying conditions (1.3). From (8.8) and (8.7) follows the inequality

$$V \geq \int_0^l \{ \varkappa_0 \sigma (u_{13}'^2 + u_{16}'^2 + u_{23}'^2 + u_{26}'^2) + \varkappa_1 \sigma (u_{12}'^2 + u_{15}'^2 + u_{21}'^2 + u_{24}'^2 +$$
$$+ u_{32}'^2 + u_{35}'^2) + \varkappa_1 (u_{31}^2 + u_{34}^2 + u_{32}^2 + u_{35}^2) + (\varkappa_0 - \Omega_0^2)(u_{13}^2 + u_{16}^2 +$$
$$+ u_{23}^2 + u_{26}^2) + (\varkappa_1 - \Omega_0^2)(u_{12}^2 + u_{15}^2 + u_{21}^2 + u_{24}^2) -$$
$$- J D^{-1} \Omega_0^2 [D_1 (u_{23} - u_{26} + u_{32} - u_{35})^2 + D_2 (u_{13} - u_{16} + u_{31} - u_{34}) +$$
$$+ 2\vartheta_{12}^\circ (u_{23} - u_{26} + u_{32} - u_{35})(u_{13} - u_{16} + u_{31} - u_{34})] \} ds \tag{8.9}$$

The Sylvester conditions for the positive definiteness of the quadratic form in the quantities u_{ij}, u_{ij}', occurring in the integrand in (8.9), reduce to the inequalities:

$$\varkappa_1 > 0, \qquad \varkappa_0 > \Omega_0^2 \tag{8.10}$$
$$(\varkappa_0 + \varkappa_1 - \Omega_0^2 - 8JD_2D^{-1}\Omega_0^2)(\varkappa_0 + \varkappa_1 - \Omega_0^2) > (\varkappa_0 - \varkappa_1 - \Omega_0^2)^2$$
$$[(\varkappa_0 + \varkappa_1 - \Omega_0^2 - 8JD_1D^{-1}\Omega_0^2)(\varkappa_0 + \varkappa_1 - \Omega_0^2) - (\varkappa_0 - \varkappa_1 - \Omega_0^2)^2] \times$$
$$\times [(\varkappa_0 + \varkappa_1 - \Omega_0^2 - 8JD_2D^{-1}\Omega_0^2)(\varkappa_0 + \varkappa_1 - \Omega_0^2) - (\varkappa_0 - \varkappa_1 - \Omega_0^2)^2] >$$
$$> [8\, J\vartheta_{12}^\circ D^{-1}\Omega_0^2\, (\varkappa_0 + \varkappa_1 - \Omega_0^2)]^2$$

From (8.6), (8.9) it follows that when conditions (8.5), (8.10) are fulfilled, $\delta^2 W_*$ is a functional positive definite in the metric $Q|_{\beta=0}$. Because W_* is continuous in the metric $Q_0|_{\beta=0}$ there follows the positive definiteness in the metric $Q|_{\beta=0}$ of functional W_*. Consequently, inequalities (8.5), (8.10) are the sufficient stability conditions for the unperturbed steady-state motion (8.1) with respect to the functionals $Q_0|_{\beta=0}$, $Q|_{\beta=0}$, T_* and P ; here in the expression for $\dot{P}$ (see Sect. 4) by ω_1, ω_2, ω_3 we should now mean the projections onto the x_1, x_2, x_3 axes of the angular velocity vector of the rigid body in its motion relative to the coordinate system $Czz'z''$. In the case when $\vartheta_{12}^\circ = 0$, conditions (8.5), (8.10) reduce to the following:

$$\varkappa_1 > 0,\ \varkappa_0 > \Omega_0^2,\ \vartheta_{33}^\circ - \vartheta_{ii}^\circ - J_{12}^{(i)} > \frac{2J\Omega_0^2(\varkappa_0 + \varkappa_1 - \Omega_0^2)}{\varkappa_1(\varkappa_0 - \Omega_0^2)} \qquad (i = 1, 2) \tag{8.11}$$

Hence we may easily obtain the stability conditions for unperturbed motion (8.1) for the cases when less than three pairs of rods are attached to the body and when the liquid is absent or wholly fills the cavity.

Thus, for example, if only one pair of rods is attached to the body, which in unperturbed motion are directed along the rotation axis, then the stability conditions are obtained from (8.11) as $\varkappa_1 \to \infty$ and have the form

$$\varkappa_0 > \Omega_0^2, \qquad \vartheta_{33}^\circ - \vartheta_{ii}^\circ - J_{12}^{(i)} > \frac{2J\Omega_0^2}{\varkappa_0 - \Omega_0^2} \qquad (i = 1, 2)$$

Analogous conditions for this case (in the absence of the liquid) were obtained in [3], however, the remarks made in Sect. 6 are valid also here. If two pairs of rods are attached to the body, perpendicular to the rotation axis in the unperturbed motion, then the stability conditions are obtained from (8.11) as $\varkappa_0 \to \infty$ and have the form

$$\varkappa_1 > 0, \qquad \vartheta_{33}^\circ - \vartheta_{ii}^\circ - J_{12}^{(i)} > \frac{2J}{\varkappa_1}\Omega_0^2 \qquad (i = 1, 2)$$

If the rods are absent, the stability conditions reduce to the following:

$$\vartheta_{33}^\circ - \vartheta_{ii}^\circ - J_{12}^{(i)} > 0 \qquad (i = 1, 2)$$

In cases when the liquid is absent or wholly fills the cavity, in the stability conditions cited we should set $J_{12}^{(1)} = J_{12}^{(2)} = 0$.

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ON THE STABILITY OF A PLAIN HOLLOW VORTEX

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Proof is given of the idifferent stability with respect to small perturbations of two flows: a hollow vortex bounded on the outside by a circular wall, and a free hollow vortex.

A method of analyzing the stability of plane potential flows of a perfect incompressible fluid with respect to small perturbations was suggested in [1] by which the difficulties arising in the determination of eigenfunctions of two-dimensional hydrodynamic flows. The method proposed there for the analysis of stability consists of the linearization of equations of hydrodynamics by conformal mapping of the unperturbed flow region onto that of the perturbed flow. It is applicable to fairly simple regions of the unperturbed flow, otherwise the feasability of conformal mapping becomes problematic. This aspect was not touched upon in [1]; some of the flows considered by the Authors cannot be analyzed in this way, since for these conformal mapping is impossible. Neither the question of completeness of the system of eigenfunctions in cases in which mapping is possible was investigated by them. It is, therefore, interesting to examine the equations arising in investigations of small perturbations of stationary flows by the method of conformal mapping, to determine its limits of applicability and, also, to solve Cauchy's problem in terms of perturbation equations.

1. A hollow vortex bounded on the outside by a circular wall. The potential flow of an incompressible fluid in the form of a plane hollow vortex bounded on the outside by a circular wall is considered. With the notation $z_0 = x_0 + iy_0$ for the complex variable in the physical plane of flow and $\zeta = u_0 - iv_0$ for the complex velocity, the flow velocity is given by formula

$$\zeta = -i\zeta^{-1}, \quad 1 \leqslant |z_0| \leqslant r^{-1}, \quad 0 < r < 1 \qquad (1.1)$$

The flow boundary $|z_0| = 1$ is free and the pressure at it is constant: $p = \text{const}$. The line $|z_0| = r^{-1}$ is the rigid wall and the flow hodograph is represented by the ring